ON SOME PROPERTIES OF FOUR CIRCLES INSCRIBED IN ONE AND CIRCUMSCRIBED ABOUT ANOTHER.

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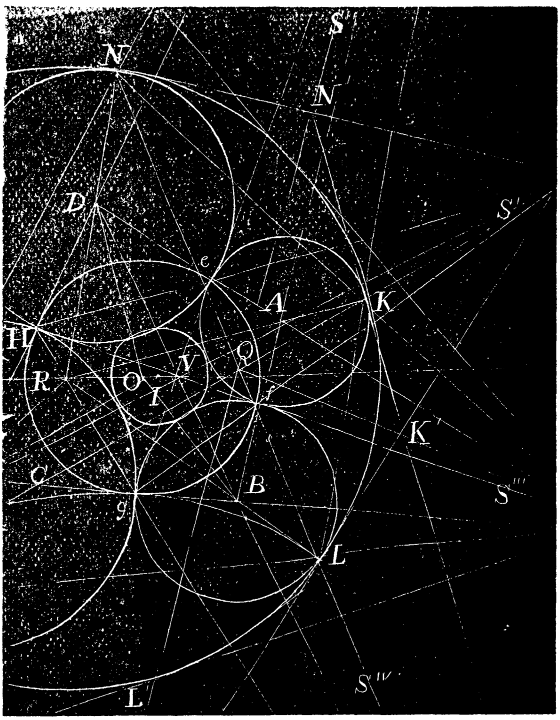
LET the circles A, B, C, D be tangent to the circle R internally and to the circle I externally.

I. The circles A, B, C, D have a common axis of similitude; for, R and I being a pair of tangent circles, their radical axis is coincident with the axis of similitude of any three, and therefore all of the circles A, B, C, D. (Salmon's *Conic Sections*, page 109, note.)



If E, F, G, H are the points of contact of these circles with each other, then HE and GF intersect in S', the external centre of similitude of the circles A and C, and HG & EF intersect in S^{iv}, the external centre of similitude of D and B.

II. The circles A, B, C, D have a common radical centre; for the perpendiculars to the sides of the quadrilateral $ABCD$ at the points E, F, G, H intersect, each two, on the line RI (Salmon, loc. cit.) and therefore all four in the same point, O. The common orthogonal circle, O, is then inscribed in $ABCD$, touching it in the points E, F, G, H.

III. Since tangents to A and D, at the points H and F respectively, meet on the radical axis of A and D (namely, in the point O), it follows that H and F are antihomologous points on the circles D and A, and hence the line HF passes through S'''. In the same way it may be shown that GE passes through S''. This is the essential quality of the quadrilaterals $ABCD$ and $EFGH$.

IV. The diagonals of $ABCD$ and $EFGH$ meet in a point, V, the polar of $S'S''$ and therefore on the line RI (Booth, *New Geometrical Methods*, Vol. II., Art. 245). This includes Problem 187 of the ANALYST.

V. If $KLMN$ is the quadrilateral formed by joining the points of contact on the circle R, its sides and diagonals pass through the points already determined on the line $S'S'''$; for N and K being centres of similitude of the circles A, D and R, NK passes through the external centre of similitude of A and D. The same is true for the corresponding circumscribed quadrilateral, $K'L'M'N'$, and the diagonals of these two quadrilaterals meet in a point, Q, which with the points $S'S^{\text{iv}}$ forms a triangle self-conjugate with respect to the circle R. In the same way it may be shown that $S'VS^{\text{iv}}$ is self-conjugate with respect to the circle I.

VI. The points S'', S', S''', S^{iv} form an harmonic range (Catalan, *The'ore'mes et Proble'mes*, III., 49).

VII. Since $BG = BL$ and $CG = CM$, L' is the centre of an exscribed circle to the triangle BRC and that circle is tangent to CB at G.

VIII. The circles about L', M', N', K' as centres, and tangent to the sides of $ABCD$, form another set of four circles circumscribed about one, the circle O, and inscriptible in another, since the quadrilaterals $L'M'N'K'$ and $LMNK$ have the diagonals of each passing through the intersections of sides of the other, and are corresponding circumscribed and inscribed quadrilaterals to the circle R (Proposition III).

The points A, B, C, D are evidently on an ellipse to which the points R and I are foci. They are concyclic only when O and V (and therefore R and I) coincide, for only then is VS''', or HF, perpendicular to EG.

IX. To inscribe in a given circle four circles tangent to each other and to another circle:—

Take any triangle, as $S'QS^{\text{iv}}$, self-conjugate with respect to the given circle, R. At K, L, M, N, the points where its sides cut the circle, R, draw tangents to R, and with their intersections as centres describe arcs orthogonal to R. From the points where the diagonals of $KLMN$ meet $S'S^{\text{iv}}$ draw tangents to these arcs; their intersections will be the centres of the required circles. The proof of this construction readily follows from the properties already obtained.

CUBIC EQUATIONS.

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IF a and b be supposed to be either positive or negative, it is well known that every cubic equation may be reduced to the form

$$x^3 - 3ax = 2b. \tag{1}$$